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THERMAL EXPLOSION OF A RISING FLOW OF LIQUID IN A RING CHANNEL

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The theory of the thermal explosion [1, 2] reduces essentially to establishing the conditions under which the medium considered, with distributed heat sources, loses thermal stability. This tradition is retained below, but we nevertheless bear in mind that thermal explosion is a consequence and occurs a certain time after the system loses thermal stability. The time interval separating the flow of liquid from the instant when it loses stability until the thermal explosion occurs, understood in the physical sense (spontaneous combustion or detonation), may turn out to be fairly long (for example, longer than the time taken for a certain technological cycle to occur). We will therefore have in mind not only the loss in stability but also the rate at which the temperature increases.

1. Formulation of the Problem. Heat is dissipated in a flow of incompressible liquid moving in an axial direction between two cylinders with fixed radii $r = r_i$, $r = r_e$, as a result of exothermic reactions and viscous dissipation. The vertical nature of the flow underlines the one-dimensional nature of the flow. We will consider the problem of determining the conditions under which an increase in the temperature of the liquid due to exothermic reactions and viscous dissipation leads to a loss in stability of the thermal state and to a thermal explosion.

The mathematical description of the thermal state of the flow of a liquid moving with constant velocity has the form [1-3]

$$\frac{\partial T}{\partial t} = \varkappa\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial x^2}\right] - v\frac{\partial T}{\partial x} + \frac{\nu}{c}\left(\frac{\partial v}{\partial r}\right)^2 + \varphi(T); \tag{1.1}$$

$$\nu\rho\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{\Delta p}{L} = 0, \tag{1.2}$$

where x and r are cylindrical coordinates, t is the time, T is the temperature, κ is the thermal diffusivity, v is the velocity of the flow, ν is the kinematic viscosity, c is the heat capacity, ρ is the density, Δp is the pressure drop taking gravitation into account, L is the length of the channel, and $\varphi(T)$ is the source function.

We will assume that the initial temperature of the liquid $T(x, r, t = 0) = T_0$, while the heat released as a result of exothermic reactions obeys Arrhenius's law

$$\varphi(T) = \frac{Qz}{\rho c}\exp\left(-\frac{E}{RT}\right).$$

Here E is the activation energy, R is the universal gas constant, Q is the thermal effect of the reaction, and z is the kinetic constant.

The solution of system (1.1) and (1.2) will be investigated for flows which satisfy the following additional conditions: a) the flow is laminar, heat transfer by conduction in the axial direction is negligibly small compared with convection, and b) the flow is piston-type flow. The stability of the thermal state of laminar flow and piston-type flow will be considered separately.

Changing to dimensionless parameters

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$$\theta = E(T - T_0)R^{-1}T_0^{-2},\ \tau = tt_a^{-1},\ x_1 = xx_a^{-1},\ r_1 = rx_a^{-1},$$
$$\beta = RT_0E^{-1},\ R_1 = r_ix_a^{-1},\ R_2 = r_ex_a^{-1},\ L_1 = Lx_a^{-1},$$
$$\upsilon_1 = \upsilon t_a x_a^{-1},\ \nu_1 = \nu E(cRT_0^2 t_a)^{-1},\ \Delta p_1 = \Delta p E(cRT_0^2\rho)^{-1}$$

($t_a = c\rho RT_0^2(EQz^{-1})\exp(E(RT_0)^{-1})$, $x_a = (\varkappa t_a)^{0.5}$ are the time scales and the distance [4]) and expansion of the function in series in powers of θ enables us to write system (1.1), (1.2) in the form

$$\frac{\partial\theta}{\partial\tau} = \frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial\theta}{\partial r_1}\right) + \frac{\partial^2\theta}{\partial x_1^2} - \upsilon_1\frac{\partial\theta}{\partial x_1}$$
$$+ \sum_{n=1}^{\infty} a_n\theta^n + \Delta\left(a_0 + \nu_1\left(\frac{\partial\upsilon_1}{\partial r_1}\right)^2\right) \equiv F(\theta, \mu, \Delta); \tag{1.3}$$

$$\nu_1\frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial\upsilon_1}{\partial r_1}\right) + \frac{\Delta p_1}{L_1} = 0, \tag{1.4}$$

where μ is a parameter from the interval containing zero, Δ is a parameter which takes the values 0 and 1, and

$$a_n = \frac{\partial^n}{\partial\theta^n}\exp(\theta(1 + \beta\theta)^{-1})\big|_{\theta=0}.$$

The theory of the stability of systems with distributed parameters [5-7] confirms that a loss in the stability of the solutions of the infinite-dimensional problem (1.3), (1.4) occurs in a space of finite dimensions.

We can reduce the dimensions of the problem in question by constructing one of its central manifolds [5] or by the projection method [6, 7]. In the latter method, the solutions of system (1.3), (1.4) are transferred to a space of finite dimensions by means of projections onto an appropriate space of eigenfunctions. By considering (1.3), (1.4) as a certain evolution problem in the space of functions constructed, we can initially study the stability of the bifurcational solution ($\Delta = 0$), and then the isolated solutions caused by the defect $\Delta = 1$.

2. Laminar Flow. By virtue of assumption a) above, the term $\partial^2\theta/\partial x_1^2$ on the right-hand side of Eq. (1.3) can be neglected, as a result of which the equation takes the form

$$\frac{\partial\theta}{\partial\tau} = \frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial\theta}{\partial r_1}\right) - \upsilon_1\frac{\partial\theta}{\partial x_1} + \Delta\left(a_0 + \nu_1\left(\frac{\partial\upsilon_1}{\partial r_1}\right)^2\right) \equiv F(\theta, \mu, \Delta). \tag{2.1}$$

We can use the following relations as the boundary and initial conditions of system (1.4), (2.1):

$$\frac{\partial\theta(x_1, R_i, \tau)}{\partial r_1} + \alpha_i\theta(x_1, R_i, \tau) = 0,\ i = 1, 2; \tag{2.2}$$

$$\upsilon_1(R_i) = 0,\ i = 1, 2; \tag{2.3}$$

$$\theta(0, r_1, \tau) = 0; \tag{2.4}$$

$$\theta(x_1, r_1, 0) = 0. \tag{2.5}$$

By integrating Eq. (1.4) with boundary conditions (2.3) we obtain the distribution of the velocities [3]

$$
\upsilon_1 = \frac{\Delta p_1 R_2^2}{4\nu_1 L_1}\left[1 - \left(\frac{r_1}{R_2}\right)^2 + \frac{1-\omega^2}{\ln(\omega^{-1})}\ln\left(\frac{r_1}{R_2}\right)\right]
$$
$$
(\omega = R_1/R_2,\ 0 < \omega \leq 1).
$$

Now, to construct the space of eigenfunctions we introduce the generating operator

$$
L_\mu\theta = \frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial\theta}{\partial r_1}\right) - \upsilon_1\frac{\partial\theta}{\partial x_1} + (a_1 - \mu)\theta \equiv \frac{\partial F(0,0,0)}{\partial\theta}\theta + \mu\frac{\partial^2 F(0,0,0)}{\partial\theta\partial\mu}\theta = 0, \tag{2.6}
$$

defined in the rectangle $(0, L_1) \times (R_1, R_2)$.

The method of separation of variables enables us to investigate the solution (2.6) in the form of a product

$$
\theta = \exp(\lambda x_1)\psi(r_1), \tag{2.7}
$$

where λ is the constant of separation.

Substituting (2.7) into (2.6) we obtain an equation in the function $\psi(r_1)$:

$$
\frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial\psi}{\partial r_1}\right) + (a_1 - \mu)\psi - \lambda\upsilon_1\psi = 0. \tag{2.8}
$$

Solution (2.7) of Eq. (2.1) with conditions (2.2), (2.4) and (2.5) in the class of exponential functions does not enable us to obtain the limits of thermal stability as a function of the channel length L_1. To obtain this relationship we will introduce the following additional constraint on the flow parameters:

$$
G = \int_0^{L_1}\left(R_2\frac{\partial\theta(x_1, R_2)}{\partial r_1} - R_1\frac{\partial\theta(x_1, R_1)}{\partial r_1}\right)dx_1 + \int_0^{L_1}\int_{R_1}^{R_2} a_1\theta(x_1, r_1)r_1 dr_1 dx_1 - \int_{R_1}^{R_2}\upsilon_1\theta(L_1, r_1)r_1 dr_1 = 0. \tag{2.9}
$$

Expression (2.9) is the linearized heat-balance equation, which establishes that, under steady-state conditions, the amount of heat arriving in the ring channel through a section $x_1 = 0$ and dissipated as a result of the reactions occurring in the flow, is equal to the amount of heat removed through the surface $r_1 = R_1$, $r_1 = R_2$ and through the section $x_1 = L_1$. This means that the increase in the mean-volume temperature leads to exothermic reactions when $\theta = 0$ and viscous dissipation is negligible, and can be neglected. A similar treatment of the heat balance is based on the fact that, due to the nonuniformity of the temperature distribution in the flow, the loss of thermal stability is of a local nature and occurs when $G = 0$, or, in other words, before the increase in the mean-volume temperature of the liquid begins. In fact, in view of the exponential form of the release of heat, acceleration of the exothermic reactions occurs over a narrow temperature range, and a loss of thermal stability should occur at the point in space at which the temperature of the liquid is a maximum.

The use of (2.9) to solve (2.1) means that the temperature distribution at the input to the channel $\theta(0, r_1, \tau)$ will be found from solution (2.7) and, consequently, will depend on the length L_1 which, of course, contradicts condition (2.4) and does not correspond to the physical process.

Pukhnachev drew attention to the fact that if L_1/R_1 is a large parameter, the contradiction between conditions (2.4) and (2.9) disappears, and (2.9) can be regarded as a consequence of the stationary form of (2.1) with conditions (2.2), (2.4) and (2.5), and the function G can be determined, apart from an additive function of the temperature $f(\theta)$, which is equal to zero when $\theta(x_1, r_1, \tau) = 0$. In fact, substituting (2.7) into (2.9) and integrating the latter with respect to x_1 we obtain an expression from which it follows, that, for large values of L_1/R_1, system (2.1), (2.2), (2.4), (2.5), and (2.9) is solvable. It follows from the same expression that if, instead of G = 0, we use G = f(θ), only a displacement of the limits of solvability of system (2.1), (2.2), (2.4), (2.5), and (2.9) with respect to the parameter L_1/R_1 occurs. This fact indicates the possibility of carrying out successive approximations of the calculations using the heat-balance equation, which takes into account the flow of energy into the channel through the section $x_1 = 0$, provided that $\theta(0, r_1\ \tau) \neq 0$ when G= 0.

Equation (2.8) with boundary conditions (2.2) is an eigenvalue problem. Of the various approximate methods available for determining the eigenvalues and eigenfunctions of such problems [8] the most formalized is the method of solution in terms of the variational calculus [8, 9]. We confirm that the function $\psi(r_1)$ is a solution of (2.8) under conditions (2.2), it the integral

$$I = \int_{R_1}^{R_2}\left[r_1\left(\frac{\partial\psi}{\partial r_1}\right)^2 + r_1(\mu - a_1)\psi^2\right]dr_1 + \sum_{i=1}^{2}\alpha_i R_i\psi^2(R_i) \tag{2.10}$$

reaches a minimum value with the constraint

$$\int_{R_1}^{R_2} v_1 r_1\psi^2 dr_1 = 1. \tag{2.11}$$

By applying Ritz's method of expansion to system (2.10), (2.11) we can write the approximate solution in the form of a series

$$\psi(r_1) = \sum_{i=1}^{N} c_i\varphi_i(r_1). \tag{2.12}$$

Here c_i are coefficients which depend on μ, and $\varphi_i(r_1)$ are any functions which satisfy boundary conditions (2.2). If we take Eq. (2.8) with $v_1 = 1$ as the basic equation, we can naturally put

$$\varphi_i(r_1) = I_0(\delta_i R_2^{-1} r_1) + \varepsilon_i N_0(\delta_i R_2^{-1} r_1), \tag{2.13}$$

where $I_n(\delta_i R_2^{-1} r_1)$ and $N_n(\delta_i R_2^{-1} r_1)$ are Bessel and Neumann functions of the first kind of the n-th order, and δ_i are the positive roots of the equation

$$\det\begin{vmatrix} \alpha_1 I_0(\delta\omega) - \delta R_2^{-1} I_1(\delta\omega) & \alpha_1 N_0(\delta\omega) - \delta R_2^{-1} N_1(\delta\omega) \\ \alpha_2 I_0(\delta) - \delta R_2^{-1} I_1(\delta) & \alpha_2 N_0(\delta) - \delta R_2^{-1} N_1(\delta) \end{vmatrix} = 0;$$

$$\varepsilon_i = \frac{\delta_i R_2^{-1} I_1(\delta_i\omega) - \alpha_1 I_0(\delta_i\omega)}{\alpha_1 N_0(\delta_i\omega) - \delta_i R_2^{-1} N_1(\delta_i\omega)}.$$

Substituting (2.12) into the integrals (2.10) and (2.11), taking (2.13) into account, and then using Lagrange's method we can obtain a system of equations in the coefficients c_i (i = 1, 2,...,N), which minimize the integral I:

$$\sum_{k=1}^{N}(\alpha_{ik} + \lambda\beta_{ik})c_k = 0 \quad (i = 1, 2,\ldots,N). \tag{2.14}$$

Here

$$\alpha_{ik} = \int_{R_1}^{R_2}\left[r_1\frac{\partial\varphi_i(r_1)}{\partial r_1}\frac{\partial\varphi_k(r_1)}{\partial r_1} + r_1(\mu - a_1)\varphi_i(r_1)\varphi_k(r_1)\right]dr_1 + \sum_{n=1}^{2}\alpha_n R_n\varphi_i(R_n)\varphi_k(R_n);\ \beta_{ik} = \int_{R_1}^{R_2} r_1 v_1\varphi_i(r_1)\varphi_k(r_1)dr_1.$$

System (2.14) has a nontrivial solution if

$$\det|\alpha_{ik} + \lambda\beta_{ik}| = 0. \tag{2.15}$$

The roots of Eq. (2.15) λ_i (i = 1, 2,..., N) are the eigenvalues of Eq. (2.8). For each λ_i we can find, from the simultaneous solution of (2.11) and (2.14), the coefficients c_k (k = 1, 2,..., N). Here it is appropriate to bear in mind that the eigenvalues λ_i and the coefficients $c_k^{(i)}$ corresponding to are functions of the parameter μ: $\lambda_i = \lambda_i(\mu)$, $c_k^{(i)} = c_k^{(i)}(\mu)$. Obviously for each λ_i there will be a certain value of μ. The maximum value is of interest from the point of view of analyzing the stability. It can be shown that it corresponds to the minimum positive eigenvalue λ_j. To do this it is necessary to substitute the eigenfuction $\psi_n(r_1)$ and integrate. The result is the relationship

$$\mu = \left[\int_{R_1}^{R_2}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial\psi_n}{\partial r_1}\right)\psi_n dr_1 + \int_{R_1}^{R_2}(a_1 - \lambda_n v_1)\psi_n^2 r_1 dr_1\right]\left[\int_{R_1}^{R_2} r_1\psi_n^2 dr_1\right]^{-1}.$$

since

$$\int_{R_1}^{R_2} r_1\psi_n^2 dr_1 > 0,\ \int_{R_1}^{R_2} r_1 v_1\psi_n^2 dr_1 > 0,$$

the maximum value of μ is reached when λ_i is a minimum.

We must consider separately the problem of the multiplicity of the eigenvalues λ_i, calculated approximately, since it is of fundamental importance when analyzing the stability by the projection method. In this respect all the solutions of (2.8) satisfy a Fredholm integral equation with an oscillating kernel, whence it follows that all the eigenvalues are simple (see [7], Appendix D). This assertion follows from the Strum oscillation theorem [8] as it applies to the system of equations (1.4), (2.1) with boundary conditions (2.2) and (2.3).

Each of the functions $\theta_i = \exp(\lambda_i x_1)\psi_i(r_1)$ is a solution of Eq. (2.1), and hence the relations $\theta = \sum_{i=1}^{N}\theta_i$, $G(\sum_{i=1}^{N}\theta_i) = 0$, hold, where, by virtue of the orthogonality of the functions $\psi_i(r_1)$ the last relation, under steady-state conditions, can be decomposed into a system of equations $G(\theta_i) = 0$ (i = 1, 2,...). Under nonstationary conditions the function θ_1 makes the main contribution to G, and the remaining solutions fall off exponentially with time.

Hence, by substituting the expression $\theta_1 = \exp(\lambda_1 x_1)\psi_1(r_1)$ into Eq. (2.9) we can calculate the value of μ as a function of the parameters of the problem β, R_1, R_2, L_1, α_1, α_2, v_1, which can be considered as bifurcational. The solution of the linearized system (1.4), (2.1) is stable if $\mu < 0$.

The space of eigenfunctions $\theta_i = \exp(\lambda_i x_1)\psi_i(r_1)$ that are orthogonal with weight $r_1 v_1$ with scalar product $<\theta_i, \theta_j^*>$ in the interval (R_1, R_2) is complete, and, consequently, any solution of nonlinear problem (1.4), (2.1), which satisfies the boundary conditions (2.2) and (2.3) can be represented with any degree of accuracy in the form of an expansion in functions of this space. The function θ_j^* (j = 1, 2,...) that appears above belongs to the operator L, conjugates with L_μ^*, and, apart from a constant factor A_j, is equal to $A_j r_1 v_1 \theta_j$.

From the definition of the region of the scalar product of eigenfunctions θ_i (i = 1, 2...) we obtain the identity

$$\langle\theta_i^n, \theta_j^*\rangle \equiv \exp((n\lambda_i + \lambda_j)x_1)\,\langle\psi_i^n(r_1), \psi_j^*(r_1)\rangle, \quad n, i, j = 1, 2,..., \tag{2.16}$$

where $\psi_i^*(r_1) = \theta_j^*\exp(-\lambda_j x_1)$.

Now, defining the amplitude ε as the projection of $\theta_0 = \theta(\mu = 0)$ on the eigenspace associated with the conjugate vector $\theta_{10}^* = \theta_1^*(\mu = 0)$: $\varepsilon = <\theta_0, \theta_{10}^*>$, we will seek the bifurcational solution ($\Delta = 0$) of system (1.4), (2.1) in the neighborhood of the point $(\theta, \mu) = (0, 0)$ in the form of the series

$$\theta = \sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}y^n,\ \mu = \sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\mu_n. \tag{2.17}$$

Here the coefficients of the expansion y_n, μ_n (n = 1, 2,...) are to be determined.

The parameter μ obtained from the equation G = 0 is a function of the length of the channel L_1, $\mu = \mu(L_1)$. By virtue of this and of the identity (2.16) the functions θ_i, θ_j^* that occur on the right-hand side of Eq. (2.17) must be found at the point $x_1 = L_1$ in the form $\bar{\theta}_i = \theta_i(L_1)$, $\bar{\theta}_j^* = \theta_j^*(L_1)$.

Substituting series (2.17) into (2.1) and equating to zero the sets of terms with like powers of the amplitude ε, we obtain the following system of equations:

$$L_0 y_1 = 0; \tag{2.18}$$

$$L_0 y_2 + 2\mu_1 \frac{\partial^2 F(0,0,0)}{\partial\theta\partial\mu} y_1 + \frac{\partial^2 F(0,0,0)}{\partial\theta^2} y_1^2 = 0 \tag{2.19}$$

and an equation in higher power of ε.

It directly follows from Eq. (2.18) that $y_1 = \bar{\theta}_{10} = \bar{\theta}_1$ ($\mu = 0$). The condition for Eq. (2.19) to be solvable

$$\langle L_0 y_2, \bar{\theta}_{10}^*\rangle = 0$$

gives for the coefficient of the expansion μ_1

$$\mu_1 = -\frac{1}{2}\left\langle \frac{\partial^2 F(0,0,0)}{\partial\theta^2}\bar{\theta}_{10}^2, \bar{\theta}_{10}^*\right\rangle \left\langle \frac{\partial^2 F(0,0,0)}{\partial\theta\partial\mu}\bar{\theta}_{10}, \bar{\theta}_{10}^*\right\rangle^{-1}.$$

Hence, the limit of stability of the solutions of system (1.4), (2.1) will be defined in the (μ, ε) plane by the expression $\mu = \mu_1\varepsilon$.

The normalization condition $<\bar{\theta}_{10}, \bar{\theta}_{10}^*> = 1$ (or, which is the same thing, $A_1 = <\bar{\theta}_{10}, r_1 v_1 \bar{\theta}_{10}>^{-1}$, $\varepsilon = 1$) enables us to write an expression for the limit of stability of the bifurcational solution of system (1.4), (2.1) in the form

$$s = \mu - \mu_1 = 0. \tag{2.20}$$

Returning to system (1.4), (2.1) with conditions (2.2) and (2.3) when $\Delta = 1$, we note that this problem differs from that considered in [10] solely in the form of the generating operator L_μ. Hence, by using the expansion of the formally introduced relationship $\Delta = \Delta(\mu, \varepsilon)$, given in [10], we can write an equation for the limit of stability of the isolated solutions when $\varepsilon = 1$:

$$s = \mu - \mu_1 + \Delta\left\langle \frac{\partial F(0,0,0)}{\partial\Delta}, \bar{\theta}_{10}^*\right\rangle\left\langle \frac{\partial^2 F(0,0,0)}{\partial\theta\partial\mu}\bar{\theta}_{10}, \bar{\theta}_{10}^*\right\rangle^{-1} = 0. \tag{2.21}$$

The limits of stability of the bifurcational and isolated solutions of system (1.4), (2.1) in the space of physical parameter β, R_1, R_2, L_1, α_1, α_2, v_1 is calculated by simultaneous solution of Eqs. (2.9), (2.20) and (2.9), (2.21), respectively.

In connection with the use of approximate methods of determining the eigen-values and eigenfuctions of the operator L_μ, the problem arises of the accuracy of the results obtained. We know [9], that as the number of terms of the series (2.12) increases the approximate eigenvalues λ_i approach the true values from above. Using the method proposed in [11], we can construct, without any essential difficulties, a space of eigenfunctions of the operator L_μ and determine the limits (2.20) and (2.21) when estimating the eigenvalues λ_i from below.

As an example, consider the flow of a liquid in a ring channel when $\omega = 0.05$. The temperature on the outer wall of the channel is kept constant ($\alpha_2 = \infty$), and the inner wall is thermally insulated ($\alpha_1 = 0$). If we confine ourselves in (2.12) to the two-term approximation (N = 2), we obtain

$$\delta_1 = 2.416,\ \delta_2 = 5.576,\ \alpha_{11} = 0.779 + 0.132R_2^2(\mu - a_1),$$
$$\beta_{11} = 6.030\cdot10^{-2}v_0R_2^2,\ \alpha_{12} = \alpha_{21} = -9.208\cdot10^{-4} + 9.968\cdot10^{-5}R_2^2(\mu - a_1),$$
$$\beta_{12} = \beta_{21} = 6.506\cdot10^{-4}v_0R_2^2,\ \alpha_{22} = 1.756 + 5.632\cdot10^{-2}R_2^2(\mu - a_1),$$
$$\beta_{22} = 2.076\cdot10^{-2}v_0R_2^2,$$

where v_0 is the maximum velocity in the flow, defined by the relation [3]

$$v_0 = \frac{\Delta p_1 R_2^2}{4\nu_1 L_1}\left[1 - \frac{1-\omega^2}{2\ln(\omega^{-1})}\left(1 - \ln\left(\frac{1-\omega^2}{2\ln(\omega^{-1})}\right)\right)\right].$$

The eigenvalue λ_i is the minimum positive root of the equation

$$\lambda^2(\beta_{11}\beta_{22} - \beta_{12}^2) + \lambda(\beta_{11}\alpha_{22} + \alpha_{11}\beta_{22} - 2\alpha_{12}\beta_{12}) + \alpha_{11}\alpha_{22} - \alpha_{12}^2 = 0.$$

The eigenfuctions of the operators L_μ, L_μ^* corresponding to the value of λ_1 when $x_1 = L_1$ have the form

$$\bar{\theta}_{10} = \exp(\lambda_1(0)L_1)c_1^{(1)}(0)\,[\varphi_1(\delta_1 R_2^{-1}r_1) + g(0)\varphi_2(\delta_2 R_2^{-1}r_1)\,],$$
$$\bar{\theta}_{10}^* = r_1 v_1 \bar{\theta}_{10}\langle\bar{\theta}_{10}, r_1 v_1 \bar{\theta}_{10}\rangle^{-1}.$$

Here

$$c_1^{(1)}(\mu) = \{v_0 R_2^2(6.030\cdot10^{-2} + 1.994\cdot10^{-4}g(\mu) + 2.076\cdot10^{-2}g^2(\mu))\}^{-0.5};$$

$$g(\mu) = \frac{c_2^{(1)}(\mu)}{c_1^{(1)}(\mu)} = -\frac{\alpha_{11} + \lambda_1(\mu)\beta_{11}}{\alpha_{12} + \lambda_1(\mu)\beta_{12}}.$$

After multiplying $\bar{\theta}_{10}^*$ by

$$(c_1^{(1)}(0))^2 R_2^3 v_0(2.583\cdot10^{-2} - 1.130\times10^{-2}g(0) + 9.958\times10^{-3}g^2(0))$$

further calculations give

$$G = c_1^{(1)}(\mu)(1 - \exp(\lambda_1(\mu)L_1))\lambda_1^{-1}(1.251 - 1.909g(\mu) - a_1 R_2^2(0.214 - 6.140\times10^{-2}g(\mu))) - c_1^{(1)}(\mu)R_2^2 v_0\exp(\lambda_1(\mu)L_1)(9.164\cdot10^{-2} - 1.769\times10^{-2}g(\mu)),$$

$$\left\langle\frac{\partial^2 F(0,0,0)}{\partial\theta^2}\bar{\theta}_{10}^2, \bar{\theta}_{10}^*\right\rangle = 2a_2(c_1^{(1)}(0))^3 R_2^3 v_0\exp(\lambda_1(0)L_1) \times (1.694\cdot10^{-2} - 2.792\times10^{-3}g(0) + 1.502\cdot10^{-2}g^2(0) - 1.545\times10^{-3}g^3(0)),$$

$$\left\langle\frac{\partial^2 F(0,0,0)}{\partial\theta\partial\mu}\bar{\theta}_{10}, \bar{\theta}_{10}^*\right\rangle = -(c_1^{(1)}(0))^2 R_2^3 v_0(2.583\times10^{-2} - 1.130\times10^{-2}g(0) + 9.958\times10^{-3}g^2(0)),$$

$$\left\langle\frac{\partial F(0,0,0)}{\partial\Delta}, \bar{\theta}_{10}^*\right\rangle = \exp(-\lambda_1(0)L_1)(4c_1^{(1)}(0)R_2\nu_1 v_0^3 \times (1.522\times10^{-2} - 1.183\times10^{-2}g(0)) + a_0 c_1^{(1)}(0)R_2^3 v_0(4.460\cdot10^{-2} + 1.995\times10^{-2}g(0))).$$

The results of calculations carried out using (2.20) and (2.21) of the limits of stability of the bifurcational and isolated solutions (curves 1 and 2) obtained with $R_2 = 6$, $\beta = 10^{-2}$, $\nu_1 = 10^{-3}$ are shown in Fig. 1. Curves 1 and 2 bound the region of stability from above.

In Fig. 2 the region of stability of the bifurcational solutions obtained with $R_2 = 6$ and $v_0 = 1$ lies between curves 1 and 2. The region of stability of the isolated solution obtained with $R_2 = 6$, $v_0 = 1$ and $\nu_1 = 10^{-3}$ lies under curve 3.

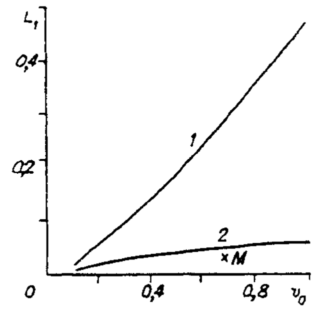

Fig. 1

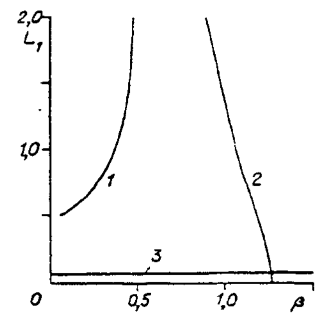

Fig. 2

We can see that the limit of stability of the isolated solutions for fixed R_2, ω, v_0, α_1, α_2, ν_1 depend very weakly on the parameter β or, for a specific liquid with specified values of E and R, on the initial temperature. Nevertheless, a change in the initial temperature of the liquid not only affects the parameter μ through $a_2(\beta)$, but also acts on the power of attraction of any solution (stable or unstable), considered as an attractor, via the time scale t_a, $\theta \sim \exp(stt_a^{-1})$. This is important in the sense that if the dimensions of the channel and the boundary conditions are such that $s > 0$ for any $\beta << 1$, then, by reducing the temperature we can achieve an increase in the scale t_a and stabilize the thermal state, meaning by this that the temperature of the liquid during a certain finite time interval will change only slightly. The problem of stabilization may become of paramount importance in the starting and stopping of technological processes connected with the transport of reactive liquid in ring channels. Thus, if the point M in Fig. 1 represents a stable thermal state with respect to curve 2, then, by aiming towards this point from $v_0 = 0$ or leaving it in the direction $v_0 = 0$, we must move in a velocity field for which the thermal state is unstable. Hence, when such processes start up or stop, when, based on the above calculations, we can assume that a loss in thermal stability is most probable, we must reduce the temperature of the liquid or, if this is possible, alter its chemical composition in such a way as to increase its activation energy. In any case, the transients must occur as rapidly as possible.

3. Piston-Type Flow. For slow flows, for which we cannot neglect the conductive component of the heat transfer along the axis, we must assume that the rate of flow is constant over the cross section of the channel $v_1 = v_0$.

In this case the generating operator is

$$L_\mu = \frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial\theta}{\partial r_1}\right) - v_0\frac{\partial\theta}{\partial x_1} + \frac{\partial^2\theta}{\partial x_1^2} + (a_1 - \mu)\theta = 0. \tag{3.1}$$

For (3.1) and for system (1.3), (1.4) boundary conditions (2.2) hold, and in the planes $x_1 = 0$, $x_1 = L_1$ they have the form

$$\theta(0, r_1, \tau) = 0, \quad \frac{\partial\theta(L_1, r_1, \tau)}{\partial x_1} = 0. \tag{3.2}$$

The eigenvalues of the operator (3.1) for boundary conditions (2.2) and (3.2) are as follows:

$$\lambda_i = a_1 - \frac{\delta_i^2}{R_2^2} - \gamma_i^2, \quad i = 1, 2, \ldots$$

Here γ_i are the positive roots of the equation $\gamma = v_0/2\mathrm{tg}(\gamma L_1)$, and the values of δ_i are the same as in (2.13). The solution of (3.1) is stable if $\mu = \lambda_1 < 0$.

All the eigenvalues λ_i are double, and to each of them there correspond two independent functions

$$\varphi_{1i} = I_0(\delta_i R_2^{-1} r_1)\sin(\gamma_i x_1)\exp(0.5 v_0 x_1),$$
$$\varphi_{2i} = N_0(\delta_i R_2^{-1} r_1)\sin(\gamma_i x_1)\exp(0.5 v_0 x_1).$$

For double eigenvalues of the operator L_μ, system (1.3), (1.4) can have three solutions for the same initial data, which represent points of intersection of conical sections in a plane with coordinates (μ_1, η). The relation between the coordinates μ_1, η and the parameters of the problem is given in [12].

The projections of the solutions of system (1.3), (1.4) on the double null-space of the operator L_μ at each point of intersection of the conical sections $(\mu_1^{(i)}; \eta^{(i)})$ (i = 1-3) can be written as

$$\theta^{(i)} = f_1^{(i)}(x_1, r_1)\exp(\mu s_1^{(i)}\tau) + f_2^{(i)}(x_1, r_1)\exp(\mu s_2^{(i)}\tau),$$

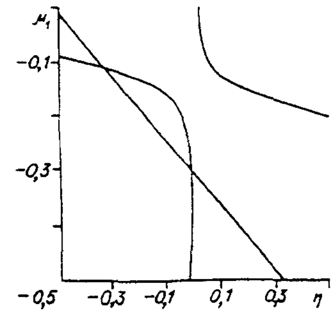

Fig. 3

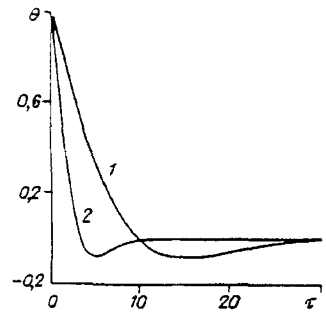

Fig. 4

where $\theta^{(i)}$ is the solution at the i-th point of intersection of the conical sections, $s_1^{(i)}$, $s_2^{(i)}$ are parameters, a method of determining which for the bifurcational and isolated solutions is described in [12], and $f_1^{(i)}(x_1, r_1)$, $f_2^{(i)}(x_1, r_1)$ are functions of the coordinates.

The parameters $s_1^{(i)}$, $s_2^{(i)}$ (i = 1-3) can be real or complex. If we consider real numbers as a special case of complex numbers, the stability of the solutions of (1.3) and (1.4) is defined by the inequality

$$\mu \max(\mathrm{Re}\, s_1^{(i)}, \mathrm{Re}\, s_2^{(i)}) < 0$$

provided that $\mathrm{Re}\, s_1^{(i)}$, $\mathrm{Re}\, s_2^{(i)}$ have the same signs [12].

We carried out calculations for a channel with walls of constant temperature $\alpha_1 = \infty$, $\alpha_2 = \infty$ for the following initial data: $\omega = 0.05$, $R_2 = 6$, $\beta = 10^{-2}$, $L_1 = 1.93$. Here and henceforth the length L_1 was chosen so that the solution of (3.1) lay on the boundary of stability ($\mu = 0$).

As a result of the calculations we obtained three bifurcational solutions of system (1.3), (1.4):

$$(\mu_1^{(1)}; \eta^{(1)}) = (1.019; -9.501),\ (\mu_1^{(2)}; \eta^{(2)}) = (-0{,}291; -1{,}912 \cdot 10^{-2}),\ (\mu_1^{(3)}; \eta^{(3)}) = (-0.113; -0.321)$$

(Fig. 3, the point $(\mu_1^{(1)}, \eta^{(1)})$ is outside the limits of the figure). These solutions are points of stationary equilibrium, for which we have the following parameters:

$$(s_1^{(1)}; s_2^{(1)}) = (-13.418; -80.242),\ (s_1^{(2)}; s_2^{(2)}) = (5.592; -1.813),\ (s_1^{(3)}; s_2^{(3)}) = (4.106; 0.929),$$

$\mu > 0$, and at the third point when $\mu < 0$. The solution at the second point is unstable for any μ.

By considering the points of stationary equilibrium as attractors, it is easy to show that both when $\mu > 0$ and when $\mu < 0$ the trapping of any perturbations of the temperature by a stable or unstable attractor is approximately equiprobable.

For the same initial data as above, but for the case when the inner wall of the channel is thermally insulated ($\alpha_1 = 0$) and $L_1 = 1.78$ (from the condition $\mu = 0$), system (1.3), (1.4) has a single real solution and two complex-conjugate solutions

$$(\mu_1^{(1)}; \eta^{(1)}) = (-0{,}227; -9{,}789),\ (\mu_1^{(2)}; \eta^{(2)}) = (-0{,}353 - 0{,}312i; 5{,}534 \cdot 10^{-2} + 0{,}446i).$$

The third point is not given. For these solutions we obtain

$$(s_1^{(1)}; s_2^{(1)}) = (0.363; -27.675),$$
$$(s_1^{(2)}; s_2^{(2)}) = (-0.150 + 0.156i; -1.649 - 2.003i),$$

whence it follows that the stationary solution is unstable for any μ, while the periodic solutions (Fig. 4, curve 1) are stable when $\mu > 0$.

An increase or decrease in the outer radius of the channel for the same initial data has no effect on the overall pattern of the distribution of the solutions, but their attractive force increases or decreases. Thus, if we take $R_2 = 12$, we have $(s_1^{(1)}; s_2^{(1)}) = (1.117; -85.177)$, $(s_1^{(2)}; s_2^{(2)}) = (-0.140 + 0.479i; -5.076 - 6.144i)$, whence it can be seen that for any μ the rate of increase or decrease of the temperature (Fig. 4, curve 2) is greater than for $R_2 = 6$.

Of course, here, as in the case of laminar flow, the rate of change of the temperature in real time must be found taking the scale t_a into account.

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REFERENCES

1. Ya. V. Zel'dovich, G. I. Barenblatt, V. B. Librovich, The Mathematical Theory of Combustion and Explosions [in Russian], Nauka, Moscow (1980).
2. D. A. Frank-Kamenetskii, Diffusion and Heat Transfer in Chemical Kinetics [in Russian], Nauka, Moscow (1987).
3. R. Bird, W. Stuart and E. Lightfoot, Transport Phenomena [Russian translation], Khimiya, Moscow (1974).
4. V. N. Vilyunov, Theory of Ignition of Condensed Materials [in Russian], Nauka, Novosibirsk (1984).
5. J. Marsden and M. MacCracken, Bifurcation of the Birth of a Cycle and its Application [Russian translation], Mir, Moscow (1980).
6. J. Yoss and D. Joseph, Elementary Theory of Stability and Bifurcation [Russian translation], Mir, Moscow (1983).
7. D. Joseph, Stability of Liquid Flow, Mir, Moscow (1981).
8. É. Kamke, Handbook of Ordinary Differential Equations [Russian translation], Fizmatgiz, Moscow (1961).
9. R. Shechter, The Variational Method in Engineering Calculations [Russian translation], Mir, Moscow (1971).
10. T. A. Bodnar', "Thermal stability of a circulating chemical reactor with a fixed layer of catalyst," Fiz. Gor. Vzryva, No. 4, 68-74 (1990).
11. S. Gould, Variational Methods in Eigenvalue Problems [Russian translation], Mir, Moscow (1970).
12. T. A. Bodnar', "The thermal stability of the tangential flow of a liquid in a ring channel," Prikl. Mek. Tekh. Fiz., No. 4, 127-133 (1991).